Probability theory can be fun and simple with dependent types (Yet another formal theory of probabilities in Coq)

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An overview of existing formalizations of probabilities in Coq [1]

InfoTheo (2009–ongoing)

- Formalizes *finite probabilities*; used for information theory [JAR 2014], error-correcting codes [JAR 2020], robust statistics [ITP 2024]

coq-proba [Tassarotti, 2023]

- Used to verify a compiler for probabilistic programming languages [PLDI 2023]

FormalML [The FormalML development team, 2023]

- Contains *advanced theorems* in probability theory, e.g., a stochastic approximation theorem [ITP 2022]

[1] Isabelle/HOL and Mathlib have extensive libraries for probabilities, this talk focuses on Coq

A proof engineering effort

Mathematical Components

29 followers https://math-comp.github.io/math-...

math-comp Public
Mathematical Components
Coq 541 109 98 34 Updated 7 minutes ago

analysis Public
Mathematical Components compliant Analysis Library
Coq 176 40 72 (1 issue needs help) 38 Updated 2 hours ago

real-closed Public
Theorems for Real Closed Fields
Coq 12 10 5 3 Updated 5 hours ago

hierarchy-builder Public
High level commands to declare a hierarchy based on packed classes
Prolog 90 MIT 19 64 13 Updated 10 hours ago

Applications of MathComp-Analysis to probabilities?

MathComp-Analysis timeline

- Asymptotic reasoning + Landau notations $\rightarrow$ differentiability [JFR 2018]
- Lebesgue integral [JAR 2023]
- Fundamental theorem of calculus [Affeldt and Stone, 2024]
- Probability theory (2023–ongoing)

Applications to probabilities

- Verified probabilistic programming languages [CPP 2023, APLAS 2023]
- Verified worst-case failure probability of real-time systems [Markovic et al., 2023]

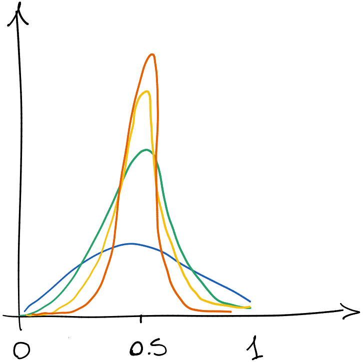
Other planned applications

- Verified robust statistics [PPDP 2021, ITP 2024]
- Verified machine learning [Ślusarz et al., ITP 2024]

An example: Bernoulli sampling [Rajani, 2019]

Bernoulli sampling

Given n independent 0-1 random variables X_i, $p \in (0, 1]$, $\theta \in (0, p)$, $\delta \in (0, 1]$ with $Pr(X_i = 1) = p$, $X = \sum_{i=1}^{n} X_i$, and $\bar{X} = \frac{X}{n}$, then $Pr(|\bar{X} - p| \leq \theta) \geq 1 - \delta$ when $n \geq \frac{3}{\theta^2} \ln(\frac{2}{\delta})$.

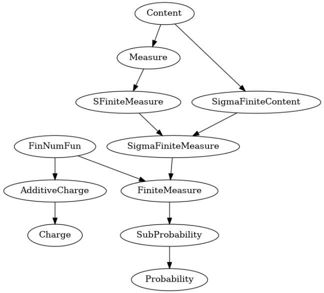


Simple and general: inherit from measure theory with Hierarchy Builder

Definition (Measure)

A measure $\mu : \mathcal{P}(T) \to \overline{\mathbb{R}}$ satisfies:

1. $\mu(\emptyset) = 0$ (measure-0)
2. $0 \leq \mu(A)$ for any set A (non-negativity)
3. $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ (σ-additivity)



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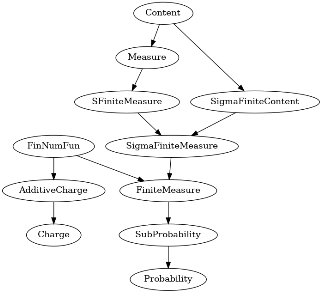

Definition (Probability measure)

A probability measure additionally implements the following interface:

```
HB.factory Record Measure_isProbability d (T : measurableType d)
    (R : realType) (P : set T -> \bar R) of isMeasure _ _ _ P :=
  { probability_setT : P setT = 1%E }.
```

…and fun: random variables and expectations

```
Context d (T : measurableType d) (R : realType) (P : probability T R).
```

Definition (Random variables)

A random variable is neither random, nor a variable. It's a measurable function from `T` to `R`.

```
Definition random_variable := {mfun T >-> R}.
Notation "{ 'RV' P >-> R }" := (@random_variable _ _ R P).
```

…and fun: random variables and expectations

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Definition (Expectation)

Expectation of X with the measure P can be expressed as the Lebesgue integral $\int X\, dP$:

```
Definition expectation (X : {RV P >-> R}) := \int[P]_w (X w)%:E.
```

Recovering discreteness

Discrete (random) variables

Discrete random variables additionally implement the following interface:

```
HB.mixin Record MeasurableFun_isDiscrete d (T : measurableType d) (R : realType)
  (X : T -> R) of @MeasurableFun d T R X := { countable_range : countable (range X) }.
```

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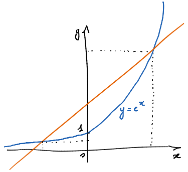
Discrete sums

When `X : {dRV P >-> R}` (the type of discrete random variables),
we build a function a_k to enumerate its values, and c_k to enumerate the probabilities,
so that the distribution can be written as $\sum_k c_k \delta_{a_k}$:

```
Lemma distribution_dRV A : measurable A ->
  distribution P X A = \sum_(k <oo) a X k * \d_(c X k) A.
```

(More) formal adventures in convex spaces

[Saikawa et al., CICM 2020] shows that probability theory benefits from a theory of *convex spaces*.
We are porting it to MathComp-Analysis to define convex functions:



Convex function

```
Definition convex_function (R : realType) (D : set R) (f : R -> R) :=
  forall t : {i01 R}, {in D &, forall (x y : R), f (x <| t |> y) <= f x <| t |> f y}.
```

Exponentials are convex

```
Lemma convex_expR : convex_function setT expR.
Lemma convex_powR p : 1 <= p -> convex_function `[0, +oo[ (fun x : R => powR x p).
```

Moments: exponential expectations

```
Definition mmt_gen_fun (X : {RV P >-> R}) (t : R) := 'E_P[expR \o t \o* X].
```

Applications of convexity: Hölder and Minkowski and L_p -spaces

We are building a theory of L_p -spaces. For that purpose we prove Hölder's and Minkowski's inequalities, which are also generally applicable to probabilities:

Hölder

```
Lemma hoelder (f g : T -> R) (p q : R) : measurable_fun setT f -> measurable_fun setT g ->
  0 < p  ->  0 < q  ->  p^-1 + q^-1 = 1 (* Hoelder conjugates *) ->
    'N_1 [f \* g] <= 'N_p [f] * 'N_q [g].
```

(Here `\+` and `\*` are pointwise addition and multiplication, and `N_p [f]` is the p-norm of `f`)

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```

Minkowski

```
Lemma minkowski f g p : measurable_fun setT f -> measurable_fun setT g -> 1 <= p ->
  'N_p%:E[f \+ g] <= 'N_p%:E[f] + 'N_p%:E[g].
```

(Here `\+` and `\*` are pointwise addition and multiplication, and `N_p [f]` is the p-norm of `f`)

More useful lemmas: Markov, Chernoff, Chebyshev and Cantelli

```
Lemma markov (X : {RV P >-> R}) (f : R -> R) (eps : R) : (0 < eps) ->
    measurable_fun [set: R] f -> (forall r, 0 <= r -> 0 <= f r) ->
    {in Num.nneg &, {homo f : x y / x <= y}} ->
  (f eps)%:E * P [set x | eps%:E <= `| (X x)%:E | ] <=
    'E_P[f \o (fun x => `| x |) \o X].

Lemma chernoff (X : {RV P >-> R}) (r a : R) : (0 < r) ->
  P [set x | X x >= a] <= mmt_gen_fun X r * (expR (- (r * a)))%:E.

Lemma chebyshev (X : {RV P >-> R}) (eps : R) : (0 < eps) ->
  P [set x | (eps <= `| X x - fine ('E_P[X])|) ] <= (eps ^- 2)%:E * 'V_P[X].

Lemma cantelli (X : {RV P >-> R}) (lambda : R) :
    P.-integrable setT (EFin \o X) -> P.-integrable setT (EFin \o (X ^+ 2)) ->
    (0 < lambda) ->
  P [set x | lambda%:E <= (X x)%:E - 'E_P[X]] <=
    (fine 'V_P[X] / (fine 'V_P[X] + lambda^2))%:E.
```

Our experiment (WIP): Bernoulli sampling [Rajani, 2019]

Theorem

Given n independent 0-1 random variables X_i, $p \in (0,1]$, $\theta \in (0,p)$, $\delta \in (0,1]$ with $Pr(X_i = 1) = p$, $X = \sum_{i=1}^{n} X_i$, and $\bar{X} = \frac{X}{n}$, then $Pr(|\bar{X} - p| \leq \theta) \geq 1 - \delta$ when $n \geq \frac{3}{\theta^2} \ln(\frac{2}{\delta})$.

becomes:

```
Theorem sampling (X : seq {RV P >-> R}) (theta delta p : R) :
  let n := size X in let X' x := ((\sum_(Xi in X) Xi) x) / n%:R in
  is_bernoulli_trial X n -> 0 < p <= 1 -> 0 < delta <= 1 ->
  0 < theta < p -> 0 < n -> 3 / theta^+2 * ln(2 / delta) <= n%:R
  -> P [set i | `| X' i - p | <= theta] >= 1 - delta%:E.
```

Conclusions

- We are generalizing Infotheo theories by porting them to MathComp-Analysis (future work: conditional probabilities, information theory, etc.)
- We are working on the verification of probabilistic programs by equational reasoning
- We aim to have a rich and general library that can be reused
- We are looking for contributors!